

MATH 303 — Measure Theory
Lecture Notes, Fall 2025

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Part 1

Motivation and Basics of Abstract Measure Theory

Motivating Problems of Measure Theory

Learning Objectives

At the end of this chapter, you will be able to:

- Compare and contrast different approaches to the “problem of measurement” in Euclidean space and identify the advantages and disadvantages of different methods
- Describe applications of measure theory to other areas of mathematics

1. The Problem of Measurement

A basic (and very old) problem in mathematics is to compute the size (length, area, volume) of geometric objects. In this chapter, we will trace the history (in a highly abbreviated form) of the mathematical developments related to the measurement of the size of geometric objects from ancient times up to the 20th century. The guide throughout will be the following open-ended questions:

- What are the “geometric objects” to which we want to (and are able to) assign a notion of size?
- What properties should size (length, area, volume) satisfy?
- How do we compute sizes of geometric objects?

This loosely-defined problem is what we will call the “problem of measurement” in Euclidean space.

2. Ancient Mathematics - Polygons, Polygons, Polygons!

In the Greek school of mathematics of antiquity¹, the computations of areas and volumes of regions was carried out by reducing general regions for which the area or volume was unknown to polygon or polyhedral regions for which the area or volume was easily computed. This consisted of two primary methods: *quadrature* (or *squaring*) and the *method of exhaustion*.

2.1. Quadrature. Quadrature (or squaring) is the process of constructing, from a given two-dimensional region, a square of equal area. This is easily carried out for simple regions such as rectangles (see Example 1.1), parallelograms, and triangles, but quickly becomes much more difficult for curved regions. The problem of “squaring the circle,” i.e. carrying out this procedure for a circle in a finite number of steps using only straightedge and compass, stumped ancient mathematicians and for good reason: the fact that π is a transcendental number (proved by Lindemann in 1882) makes a solution to the problem impossible.

EXAMPLE 1.1: QUADRATURE OF A RECTANGLE

Given a rectangle with sides a, b , we can square the rectangle as follows. Place segments of length a and b end to end and form a (semi)circle with diameter given by the two segments

¹A very enlightening discussion of the history of “Greek mathematics” recently took place in the pages of the *Notices of the American Mathematical Society*; see [Kim25, Net25].

(of total length $a + b$), and draw a segment perpendicular to the diameter at the meeting point of the two segments (see Figure 1.1).

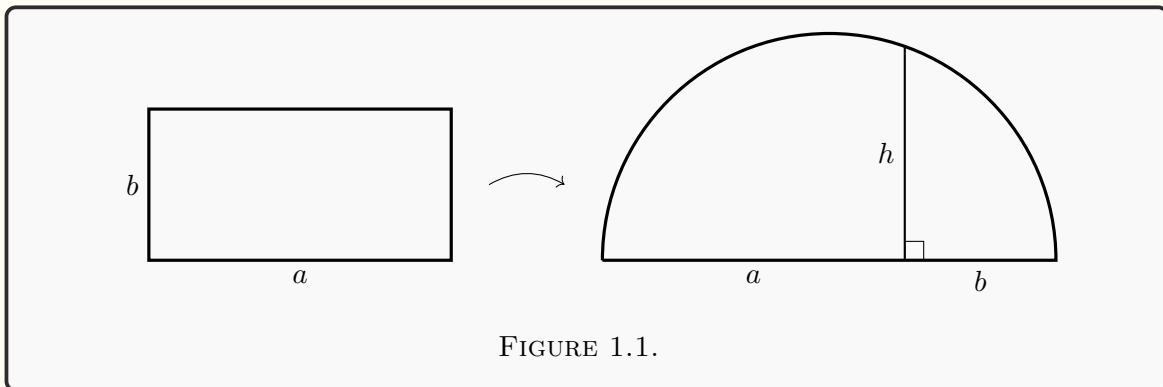


FIGURE 1.1.

To compute the height h , form two right triangles over bases a and b (see Figure 1.2). The two hypotenuses meet at a right angle by Thales's theorem. Then by three applications of the Pythagorean theorem,

$$\underbrace{(a + b)^2}_{a^2 + b^2 + 2ab} = \underbrace{(a^2 + h^2) + (b^2 + h^2)}_{a^2 + b^2 + 2h^2},$$

whence $h^2 = ab$. The square with base h shown in red thus has the same area as the original rectangle, so we have successfully squared the rectangle.

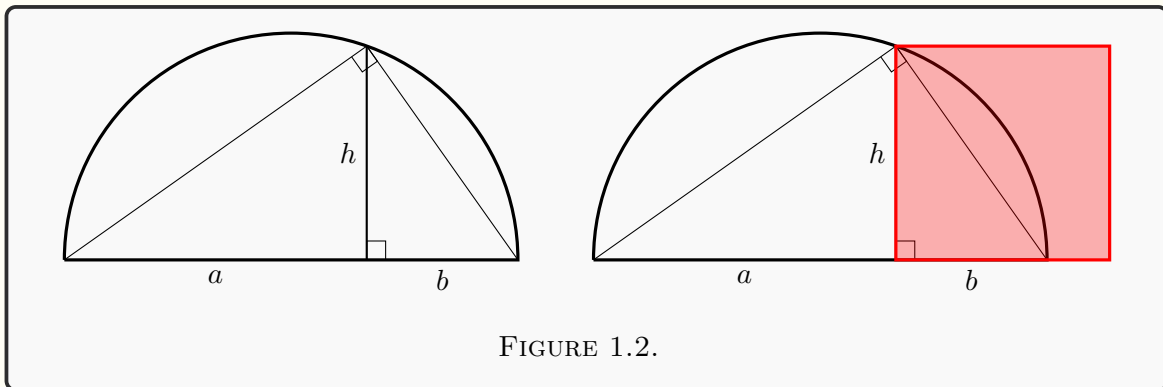
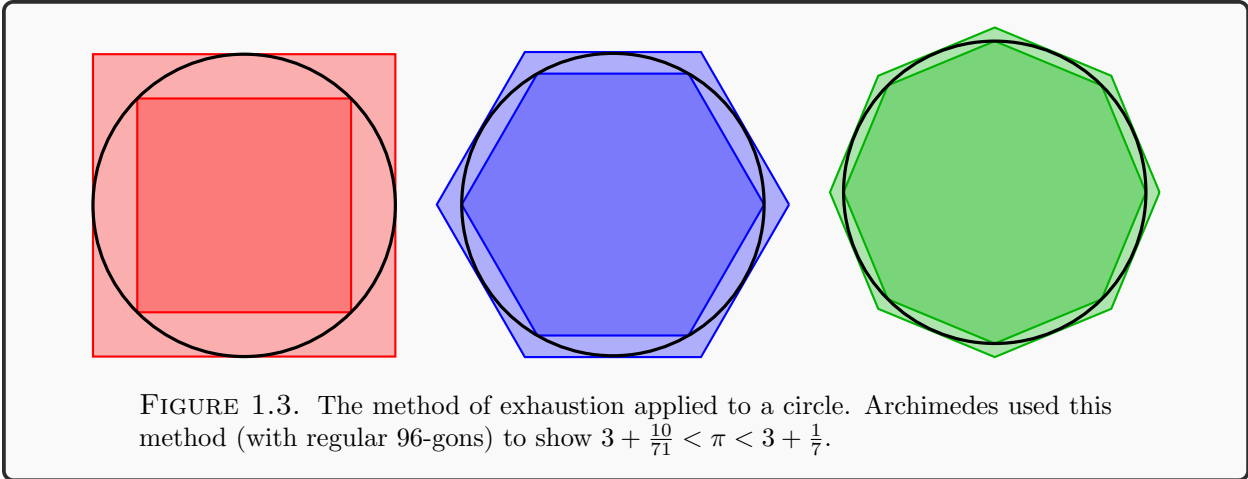


FIGURE 1.2.

2.2. Method of Exhaustion. Another method utilized in antiquity and containing the seeds of later developments in analysis was the method of exhaustion. Credited to Eudoxus for establishing the method rigorously, the method of exhaustion consists of inscribing and circumscribing sequences of polygons that converge to a given shape (see Figure 1.3). When used in conjunction with the method of squaring, which can be used to compute polygonal areas, the method of exhaustion was a powerful method for measurement in Greek mathematics. Polygonal approximations (rediscovered and improved in various locations and times) continued to be the best-known method for computing π until the end of the 17th century.



3. Indivisibles and Infinitesimals

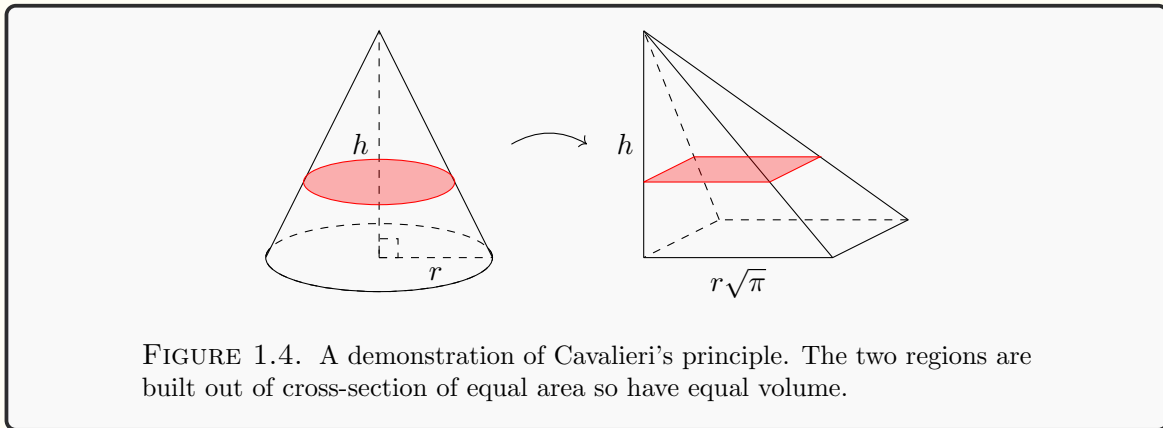
A significant breakthrough in the computation of areas and volumes was formalized in the 17th century by Bonaventura Cavalieri. (Similar methods had been used in antiquity by Archimedes and in the 5th century CE by Chinese mathematicians Zu Chongzhi and Zu Gengzhi, but the method was not in common usage in early modern Europe.) Cavalieri's principle can be expressed as follows:

Suppose two regions in the plane are bounded between two parallel lines. If the two regions have cross-sections of equal length, then they have equal area.

A corresponding statement also holds in three dimensions: if two regions have planar cross-sections of equal area, then they have equal volume.

EXAMPLE 1.2

Cavalieri's principle can be used to compute the volume of a cone. First, by slicing parallel to the base, Cavalieri's principle shows that the volume of a pyramidal region depends only on the area of the base and the height. In particular, the computation of the volume of a cone (or any other starting pyramid) can be reduced to computing the volume of a square pyramid (Figure 1.4).



The volume of the resulting pyramid can be computed by observing that three pyramids for which the height is equal to the side length of the base can be combined into a cube (Figure 1.5).

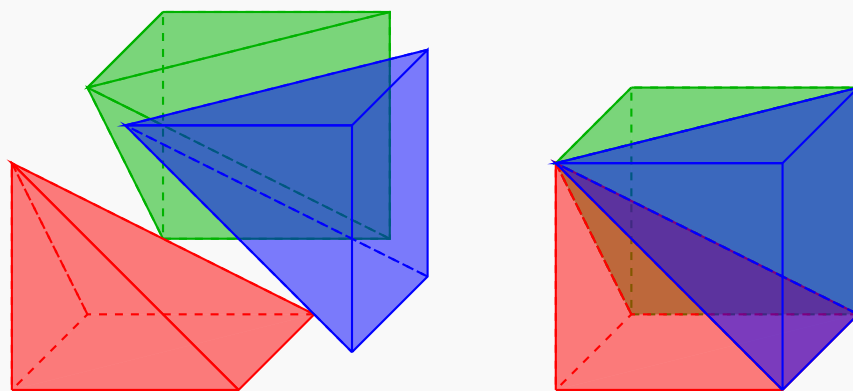


FIGURE 1.5. Three square pyramids combine to form a cube, so the volume of each pyramid is one third the volume of the cube.

Since the volume of the pyramid scales proportionally to its height, we conclude that the pyramid (and hence the cone) has volume $\frac{1}{3}\pi r^2 h$.

While the volume of a cone was known in antiquity using the method of exhaustion (it appeared, for example, in the 12th book of Euclid’s *Elements*), Cavalieri’s principle provides a much simpler proof. As a result of the negative resolution of Hilbert’s third problem, it is also now known that the computation of the volume of a cone or even certain pyramids requires some form of infinitary argument (using Cavalieri’s indivisibles, integral calculus, or a limiting process as in the method of exhaustion), since polyhedra of equal volume cannot always be transformed into each other via finitely many polyhedral cuts and rearrangements.

4. Integral Calculus

Notwithstanding earlier developments from Menaechmus and Apollonius in ancient Greek mathematics and Omar Khayyam in 11th century Persian mathematics, the introduction of coordinate systems by Descartes in the 17th century set forth the discipline of *analytic geometry* and revolutionized geometric calculations by uniting geometry with algebra. The variety of “geometric objects” was no longer limited to polygons, polyhedra, conic sections, and other classical objects; mathematicians had been unleashed to describe an endless assortment of new shapes by means of algebraic formulae. But how was one to compute their sizes?

Following earlier contributions by Cavalieri (who computed the area under $y = x^n$ for $n \leq 9$), Wallis (who extended Cavalieri’s work to general $n \in \mathbb{Z}$), and many others, Newton and Leibniz discovered an astonishing link between the computation of areas (integration) and differentiation, namely the *fundamental theorem of calculus*.

5. Introducing ε and δ

Early work in calculus was based on infinitesimals and does not meet our present-day standards for mathematical rigor. Though a rigorous foundation for the theory of infinitesimals was eventually established by Abraham Robinson in the 1960s (dubbed “nonstandard analysis” as a result of its later historical development), calculus was first put on firm foundations by the “ ε - δ ” formalism established in the 19th century by Cauchy, Weierstrass, and others. Using the newly rigorous notions of limits, the ancient method of exhaustion could finally reach its full potential with the integration theory developed by Riemann and Darboux. For purposes of exposition, we

will focus on Darboux's approach to integration, which is very similar to Riemann's but with some simplifications.

DEFINITION 1.3: DARBOUX INTEGRATION

Let $B = \prod_{i=1}^d [a_i, b_i]$ be a closed box in \mathbb{R}^d , and let $f : B \rightarrow \mathbb{R}$ be a bounded function.

- A *Darboux partition* of B is a family of finite sequences $(x_{i,j})_{1 \leq i \leq d, 0 \leq j \leq n_i}$ such that $a_i = x_{i,0} < x_{i,1} < \dots < x_{i,n_i} = b_i$ for each $i \in \{1, \dots, d\}$.

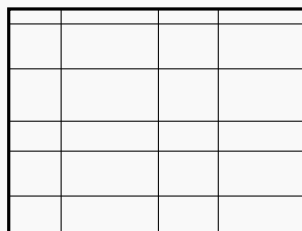


FIGURE 1.6. A Darboux partition in dimension $d = 2$ with $n_1 = 4$ and $n_2 = 6$.

- Given a Darboux partition $P = (x_{i,j})_{1 \leq i \leq d, 0 \leq j \leq n_i}$ of B , the *upper* and *lower Darboux sums of f over B* are given by

$$U_B(f, P) = \sum_{\mathbf{j} \in \prod_{i=1}^d \{1, \dots, n_i\}} \sup_{\mathbf{x} \in B_{\mathbf{j}}} f(\mathbf{x}) \cdot \text{Vol}(B_{\mathbf{j}})$$

and

$$L_B(f, P) = \sum_{\mathbf{j} \in \prod_{i=1}^d \{1, \dots, n_i\}} \inf_{\mathbf{x} \in B_{\mathbf{j}}} f(\mathbf{x}) \cdot \text{Vol}(B_{\mathbf{j}}),$$

where $B_{\mathbf{j}}$ is the box $\prod_{i=1}^d [x_{i,j_i-1}, x_{i,j_i}]$, and $\text{Vol}(B_{\mathbf{j}}) = \prod_{i=1}^d (x_{i,j_i} - x_{i,j_i-1})$ is the volume of $B_{\mathbf{j}}$.

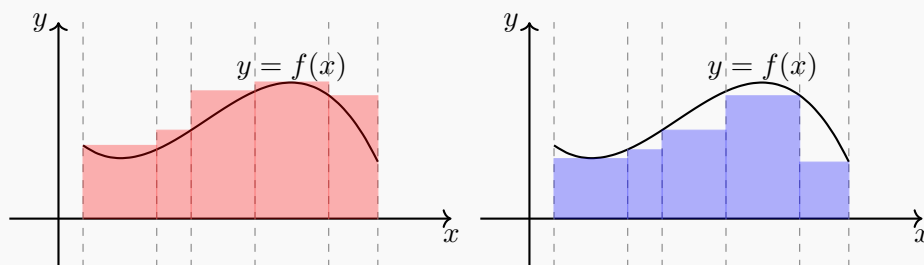


FIGURE 1.7. Upper (red) and lower (blue) Darboux sums of a function f over an interval ($d = 1$).

- The *upper* and *lower Darboux integral of f over B* are

$$U_B(f) = \inf \{ U_B(f, P) : P \text{ is a Darboux partition of } B \}$$

and

$$L_B(f) = \sup\{L_B(f, P) : P \text{ is a Darboux partition of } B\}.$$

- The function f is *Darboux integrable over B* if $U_B(f) = L_B(f)$, and their common value is called the *Darboux integral of f over B* and is denoted by $\int_B f(\mathbf{x}) \, d\mathbf{x}$.

Since Riemann integration is more commonly taught, we mention that the Darboux integral and the Riemann integral define the same quantity.

PROPOSITION 1.4

A function f is Darboux integrable if and only if it is Riemann integrable. Moreover, the value of the Darboux integral and the Riemann integral (for a Riemann–Darboux integrable function) are the same.

Because of its flexibility in terms of the dimension of the ambient Euclidean space, the Riemann–Darboux integral comes with an attendant notion of size or “hyper-volume” for regions in Euclidean space: the *Jordan content*.

DEFINITION 1.5

A bounded set $E \subseteq \mathbb{R}^d$ is a *Jordan measurable set* if $\mathbb{1}_E$ is Riemann–Darboux integrable over a box containing E . The *Jordan content* of a Jordan measurable set E is the value $J(E) = \int_B \mathbb{1}_E(\mathbf{x}) \, d\mathbf{x}$, where B is any closed box containing E .

Jordan measurable sets include basic geometric objects such as polyhedra, conic sections, regions bounded by finitely many smooth curves/surfaces, etc. The basic building blocks for Jordan measurable sets are what are called *simple sets* (or *elementary sets*).

DEFINITION 1.6

An *interval* in \mathbb{R} is a set of the form (a, b) , $[a, b)$, $(a, b]$, or $[a, b]$ for some real numbers $a \leq b$. A *box* in \mathbb{R}^d is a set of the form $B = \prod_{i=1}^d I_i$, where I_1, \dots, I_d are intervals. A set $S \subseteq \mathbb{R}^d$ is a *simple set* (or *elementary set*) if it is a finite union of boxes $S = \bigcup_{j=1}^k B_j$.

If the boxes B_1, \dots, B_k are disjoint, then the volume of the simple set $S = \bigcup_{j=1}^k B_j$ is $\text{Vol}(S) = \sum_{j=1}^k \text{Vol}(B_j)$. If some of the boxes intersect, then the volume of $S = \bigcup_{j=1}^k B_j$ can be computed using inclusion-exclusion:

$$\text{Vol}(S) = \sum_{j=1}^k \text{Vol}(B_j) - \sum_{1 \leq j_1 < j_2 \leq k} \text{Vol}(B_{j_1} \cap B_{j_2}) + \sum_{1 \leq j_1 < j_2 < j_3 \leq k} \text{Vol}(B_{j_1} \cap B_{j_2} \cap B_{j_3}) - \dots$$

This expression is well-defined, since the intersection of two boxes is again a box. A Jordan measurable set is a set that is “well-approximated” by simple sets, as we will make precise now.

DEFINITION 1.7

For a bounded set $E \subseteq \mathbb{R}^d$, define the *inner* and *outer Jordan content* by

$$J_*(E) = \sup \{ \text{Vol}(S) : S \subseteq E \text{ is a simple set} \}.$$

and

$$J^*(E) = \inf \{ \text{Vol}(S) : S \supseteq E \text{ is a simple set} \}.$$

The inner Jordan content can be viewed as a generalization of the method of approximation by inscribed polygons and the outer Jordan content as a generalization of the method of approximation by circumscribed polygons. In order to make sense of the size of an object using the Jordan content, the inscribed and circumscribed regions must approach the same size. In other words, Jordan measurable sets are those for which this extended method of exhaustion successfully converges. This is made precise by the following theorem.

THEOREM 1.8

Let $E \subseteq \mathbb{R}^d$ be a bounded set. The following are equivalent:

- (i) E is Jordan measurable;
- (ii) $J_*(E) = J^*(E)$ (in which case $J(E)$ is equal to this same value);
- (iii) $J^*(\partial E) = 0$.

We do not include a proof of Theorem 1.8 but indicate its core content in Figure 1.8.

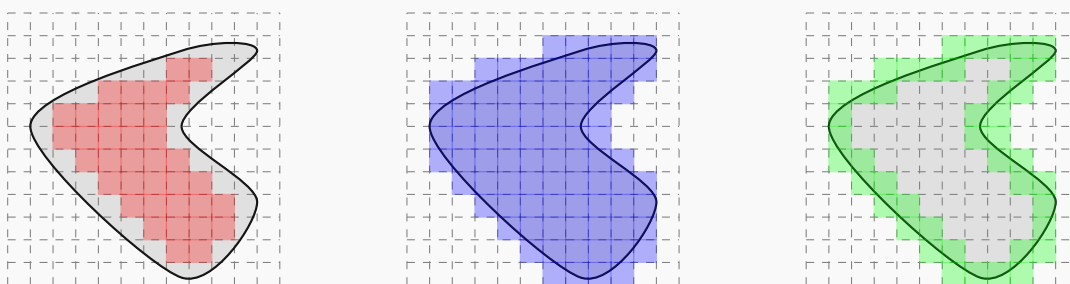


FIGURE 1.8. Simple sets approximating the inner (red) and outer Jordan content (blue) of a region in dimension $d = 2$. With the red boxes removed from the blue, we get a simple set covering the boundary (in green).

While the family of Jordan measurable sets is quite vast, a consequence of Theorem 1.8 is that not all sets are Jordan measurable.

EXAMPLE 1.9

The sets $\mathbb{Q} \cap [0, 1]$ and $[0, 1] \setminus \mathbb{Q}$ are not Jordan measurable.

In addition to the above example, there are many other “nice” sets that are not Jordan measurable. There are, for instance, bounded open sets in \mathbb{R} that are not Jordan measurable. We will work out one such example in detail.

EXAMPLE 1.10

The complement U of the fat Cantor set (also known as the Smith–Volterra–Cantor set) $K \subseteq [0, 1]$ is Jordan non-measurable. We construct K iteratively, starting from $[0, 1]$, by

removing intervals of length 4^{-n} at step n . In other words, at step n , we remove an interval of length 4^{-n} around each rational point with denominator 2^n .

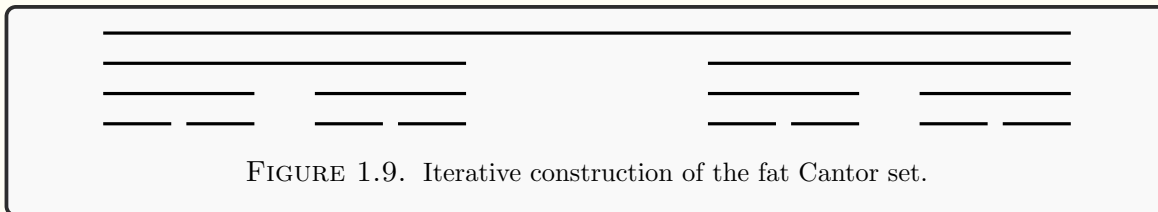


FIGURE 1.9. Iterative construction of the fat Cantor set.

Let

$$U = \bigcup_{n=0}^{\infty} \bigcup_{j=1}^{2^n} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right).$$

Then $K = [0, 1] \setminus U$. The inner Jordan content of U is

$$J_*(U) = \sum_{n=0}^{\infty} \sum_{j=1}^{2^n} \text{Len} \left(\frac{2j+1}{2^{n+1}} - \frac{1}{2 \cdot 4^{n+1}}, \frac{2j+1}{2^{n+1}} + \frac{1}{2 \cdot 4^{n+1}} \right) = \sum_{n=0}^{\infty} 2^n \cdot \frac{1}{4^{n+1}} = \frac{1}{4} \sum_{n=0}^{\infty} 2^{-n} = \frac{1}{2}.$$

However, $\bar{U} = [0, 1]$ (since U contains every rational number whose denominator is a power of 2), so the outer Jordan content of U is $J^*(U) = J^*([0, 1]) = 1$.

6. Set Theory, Choice, and the Impossibility of Measuring Everything

As mathematicians continued the quest for formalization, the notion of “geometric object” continued to expand in possible meaning. With set theory taking its place at the foundations of mathematics, one could now dream of perhaps assigning a size to arbitrary subsets of Euclidean space. Giuseppe Vitali dashed such hopes with a clever construction in 1905.

Thus far, our discussion of the notion of size has been largely based on geometric intuition. In order to say that there exists a set incapable of being assigned a sensible notion of size, we now take an axiomatic approach and reformulate (a version of) the problem of measurement as a concrete mathematical statement.

PROBLEM 1.11: THE PROBLEM OF MEASUREMENT (STRONG FORM)

Let $d \in \mathbb{N}$. Does there exist a notion of d -dimensional volume Vol , defined for all subsets of \mathbb{R}^d , such that

- NORMALIZED: $\text{Vol}([0, 1]^d) = 1$;
- ISOMETRY-INVARIANT: if A and B are isometric, then $\text{Vol}(A) = \text{Vol}(B)$;
- COUNTABLY ADDITIVE: if E_1, E_2, \dots are pairwise disjoint, then $\text{Vol}(\bigsqcup_{n \in \mathbb{N}} E_n) = \sum_{n=1}^{\infty} \text{Vol}(E_n)$.

Vitali showed that Problem 1.13 has a negative answer even for $d = 1$.

THEOREM 1.12

There is no normalized, translation-invariant, countably additive function defined for all subsets of \mathbb{R} .

PROOF. Define an equivalence relation on $[0, 1)$ by $x \sim y$ if $y - x \in \mathbb{Q}$. By the axiom of choice, let $E \subseteq [0, 1)$ be a set containing exactly one representative of each equivalence class. For each $t \in \mathbb{Q} \cap [0, 1)$, let $E_t = \{x + t \bmod 1 : x \in E\} \subseteq [0, 1)$.

CLAIM 1. The sets $(E_t)_{t \in \mathbb{Q} \cap [0, 1)}$ are pairwise disjoint.

For $t, s \in \mathbb{Q} \cap [0, 1)$ and $x, y \in E$, if $x + t \equiv y + s \pmod{1}$, then

$$y - x \equiv t - s \pmod{1},$$

so $x \sim y$. But E contains only one element from each equivalence class, so $x = y$ and $t = s$.

CLAIM 2. $\bigsqcup_{t \in \mathbb{Q} \cap [0, 1)} E_t = [0, 1)$

Let $x \in [0, 1)$. Then there exists $y \in E$ with $y \sim x$, since E has a representative of each equivalence class. Let $t = x - y \bmod 1 \in \mathbb{Q} \cap [0, 1)$. Then

$$y + t \equiv x \pmod{1}.$$

so $x \in E_t$.

Suppose for contradiction that L is a normalized, translation-invariant, countably additive function defined for all subsets of \mathbb{R} .

CLAIM 3. For every $t \in \mathbb{Q} \cap [0, 1)$, $L(E_t) = L(E)$.

We can write

$$E_t = ((E + t) \cap [0, 1)) \sqcup ((E + t) \cap [1, 2) - 1).$$

Therefore, by translation-invariance,

$$L(E_t) = L(E + t) = L(E).$$

Combining Claims 1–3 and using countable additivity of L ,

$$1 = L([0, 1)) = \sum_{t \in \mathbb{Q} \cap [0, 1)} L(E_t) = \sum_{t \in \mathbb{Q} \cap [0, 1)} L(E) = \infty \cdot L(E).$$

But there is no value of $L(E)$ that can satisfy this equation, so we have reached a contradiction. \square

Confronted with Vitali's example, one must make some compromise. In order to comport with an intuitive meaning of "size," normalization and isometry-invariance appear absolutely essential. This leaves two options: (1) restrict the domain of the volume function to only assign size to a certain subclass of "nice" sets and hope to avoid the pathologies of the Vitali sets, or (2) sacrifice countable additivity for the weaker notion of finite additivity. We address the two possibilities in turn, starting with the latter. Relaxing our additivity assumption to *finite additivity*, we arrive at a new form of the problem of measurement.

PROBLEM 1.13: THE PROBLEM OF MEASUREMENT (WEAK FORM)

Let $d \in \mathbb{N}$. Does there exist a notion of d -dimensional volume Vol , defined for all subsets of \mathbb{R}^d , such that

- NORMALIZED: $\text{Vol}([0, 1]^d) = 1$;
- ISOMETRY-INVARIANT: if A and B are isometric, then $\text{Vol}(A) = \text{Vol}(B)$;
- FINITELY ADDITIVE: if A and B are disjoint, then $\text{Vol}(A \sqcup B) = \text{Vol}(A) + \text{Vol}(B)$.

Surprisingly, the solvability of this weak form of the problem of measurement depends on the dimension d . The problem was solved by Banach in dimensions $d = 1$ and $d = 2$ using a version of the Hahn–Banach theorem from functional analysis. However, in dimensions 3 and higher, a paradoxical situation emerges.

THEOREM 1.14: BANACH–TARSKI THEOREM

Let $d \geq 3$. Given any two bounded regions $A, B \subseteq \mathbb{R}^d$, both with nonempty interior, there exist partitions $A = A_1 \sqcup \dots \sqcup A_k$ and $B = B_1 \sqcup \dots \sqcup B_k$ for some $k \in \mathbb{N}$ such that A_i and B_i are congruent for each $i \in \{1, \dots, k\}$. In particular, the unit ball can be decomposed into finitely many pieces and reassembled into two congruent copies of the unit ball.

Thus, at least in high dimensional situations, even with weakened the notion of “size” to only be finitely additive, there is no consistent way to measure every subset of \mathbb{R}^d . There is also good reason to insist on the property of countable additivity. For example, finitely-additive notions of measure produce integrals that do not interact with limits in the way that one might hope. For ease of exposition, we give an example with the Riemann integral, but similar examples can be constructed for any notion of integration that fails to be countably additive (including Banach’s notion of integration in dimensions 1 and 2).

EXAMPLE 1.15

Enumerate the set $\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \dots\}$. Let $f_n : [0, 1] \rightarrow [0, 1]$ be the function

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \{q_1, \dots, q_n\} \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n is Riemann integrable and $f_n \rightarrow \mathbb{1}_{\mathbb{Q} \cap [0, 1]}$ pointwise, but $\mathbb{1}_{\mathbb{Q} \cap [0, 1]}$ is not Riemann integrable.

Gathering all of our observations thus far, the best we can hope for in addressing the problem of measurement is to exhibit a rich class of “measurable sets” (including, for example, polygons, circles, open sets², and general Jordan-measurable sets) for which we can define a normalized, translation-invariant, and countably additive notion of measure.

7. The Solution of Lebesgue

The Jordan non-measurable set in Example 1.10 appears to have a sensible notion of “length.” Indeed, the complement U , being a disjoint union of intervals, could be reasonably assigned as a “length” the sum of the lengths of the (countably many) intervals of which it is made. This produces a value of $\frac{1}{2}$ for the length of U , and so we should take K to also have length $\frac{1}{2}$, since $K \sqcup U = [0, 1]$ is an interval of length 1. The feature that U is a disjoint union of intervals turns out to not be any special feature of U at all but instead a general feature of open sets in \mathbb{R} .

²Example 1.10 shows that there are open sets that are not Jordan-measurable, so we need a more general construction to handle arbitrary open sets.

PROPOSITION 1.16

Let $U \subseteq \mathbb{R}$ be an open set. Then U can be expressed as a countable disjoint union of open intervals.

By Proposition 1.16, it seems reasonable to define the length of an open set $U \subseteq \mathbb{R}$ as follows. Write $U = (a_1, b_1) \sqcup (a_2, b_2) \sqcup \dots$ as a disjoint union of open intervals, and define its length as $(b_1 - a_1) + (b_2 - a_2) + \dots$. Then open sets may play the role that simple sets played in the definition of the Jordan content, and this leads to the Lebesgue measure.

REMARK. In higher dimensions, Proposition 1.16 needs to be modified, but one can still reasonably talk about the d -dimensional volume of open sets in \mathbb{R}^d . The key is to replace open intervals with half-open boxes $\prod_{i=1}^d [a_i, b_i)$.

DEFINITION 1.17

Let $E \subseteq \mathbb{R}^d$.

- The *outer Lebesgue measure of E* is the quantity

$$\begin{aligned} \lambda^*(E) &= \inf \{ \text{Vol}(U) : U \supseteq E \text{ is open} \} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \text{Vol}(B_j) : B_1, B_2, \dots \text{ are boxes, and } E \subseteq \bigcup_{j=1}^{\infty} B_j \right\}. \end{aligned}$$

- The set E is *Lebesgue measurable* (with *Lebesgue measure* $\lambda(E) = \lambda^*(E)$) if for every $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ such that $E \subseteq U$ and $\lambda^*(U \setminus E) < \varepsilon$.

PROPOSITION 1.18

If $E \subseteq \mathbb{R}^d$ is Jordan measurable, then E is Lebesgue measurable and $J(E) = \lambda(E)$.

The family of Lebesgue measurable sets is much larger than the family of Jordan measurable sets. Among the several nice properties of the Lebesgue measure (and abstract measures) that we will see later in the course are:

PROPOSITION 1.19

- (1) If $(E_n)_{n \in \mathbb{N}}$ are Lebesgue measurable sets, then $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$ are Lebesgue measurable.
- (2) If $(E_n)_{n \in \mathbb{N}}$ are pairwise disjoint and Lebesgue measurable, then $\lambda(\bigsqcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$.
- (3) if A and B are congruent, then $\lambda(A) = \lambda(B)$.
- (4) If $E_1 \subseteq E_2 \subseteq \dots \subseteq \mathbb{R}^d$ are Lebesgue measurable sets, then $\lambda(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \lambda(E_n)$.
- (5) If $E_1 \supseteq E_2 \supseteq \dots$ are Lebesgue measurable subsets of \mathbb{R}^d and $\lambda(E_1) < \infty$, then $\lambda(\bigcap_{n=1}^{\infty} E_n) = \lim_{n \rightarrow \infty} \lambda(E_n)$.

The “continuity” properties expressed in items (4) and (5) result in a corresponding notion of integration that is able to interact desirably with pointwise limits, overcoming the shortcomings of the Riemann integral illustrated in Example 1.15.

8. Applications of Abstract Measure Theory

The mathematical language and tools encompassed in measure theory play a foundational role in many other areas of mathematics. A highly abbreviated sampling follows.

PROBABILITY THEORY. Measure theory provides the axiomatic foundations of probability theory, providing rigorous notions of *random variables* and *probabilities of events*. Important limit laws (the law of large numbers and central limit theorem, for example) are phrased mathematically using measure-theoretic notions of convergence.

FOURIER ANALYSIS. Periodic (say, continuous or Riemann-integrable) functions on the real line have corresponding Fourier series representations $f(x) \sim \sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$. The functions $e^{2\pi i n x}$ are orthonormal, and Parseval's identity gives $\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 = \int_0^1 |f(x)|^2 dx$. Given a sequence $(a_n)_{n \in \mathbb{N}}$, one may ask whether $\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n x}$ is the Fourier expansion of some function f , and if so, what properties does f have? Another natural question is whether the series $\sum_{n \in \mathbb{Z}} \widehat{f}(n) e^{2\pi i n x}$ actually converges to the function f , and if so, in which sense? Both of these questions are properly answered in a measure-theoretic framework. If one is interested in decomposing functions defined on other groups (for instance, on compact abelian groups) into their Fourier series, then one also needs to develop a method of integrating functions on groups in order to compute Fourier coefficients and make sense of Parseval's identity.

FUNCTIONAL ANALYSIS AND OPERATOR THEORY. When one studies familiar concepts from linear algebra in infinite-dimensional spaces, measures become unavoidable for many tasks. For example, versions of the spectral theorem (generalizing the representation of suitable matrices in terms of their eigenvalues and eigenvectors) for operators on infinite-dimensional spaces require the abstract notion of a measure.

ERGODIC THEORY. Ergodic theory was developed to study the long-term statistical behavior of dynamical (time-dependent) systems, providing a framework to resolve important problems in physics related to the “ergodic hypothesis” in thermodynamics and the “stability” of the solar system. It turns out that the appropriate mathematical formalism for understanding these problems comes from abstract measure theory.

FRactal Geometry. Self-similar geometric objects such as the Koch snowflake, Sierpiński carpet, and the middle-thirds Cantor set (see Figure 1.10) can be meaningfully assigned a notion of “dimension” that can take a non-integer value. How does one determine the dimension of a fractal object? There are several different approaches to dimension, but one of the most popular is the *Hausdorff dimension*, which relies on a family of measures that interpolate between the integer-dimensional Lebesgue measures.

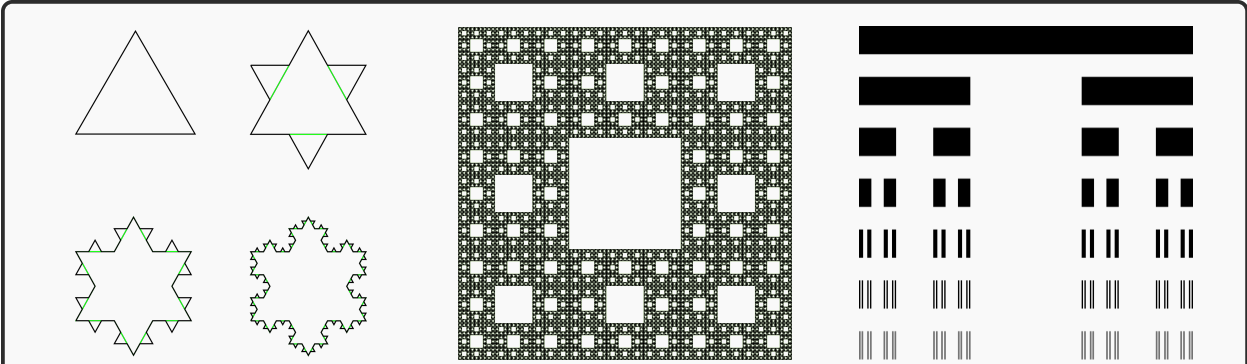


FIGURE 1.10. Fractal shapes: the Koch snowflake (left) of Hausdorff dimension $\frac{\log 4}{\log 3} \approx 1.26$, Sierpiński carpet (middle) of dimension $\frac{\log 8}{\log 3} \approx 1.89$, and middle-thirds Cantor set (right) of dimension $\frac{\log 2}{\log 3} \approx 0.63$.

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